Topic 4: Elementary Probability

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Basic Probability Concepts

- Probability: a numerical value that represents the chance, likelihood, possibility that an event will occur (always between 0 and 1)
- Event: each possible outcome of a variable
- Impossible Event: an event that has no chance of occurring (probability $= 0$)
- Certain Event: an event that is sure to occur (probability $= 1$)

Assessing Probability

There are three approaches to assessing the probability of an uncertain event:

- **a priori probability**
  - based on prior knowledge
  - probability of occurrence $= \frac{X}{T} = \frac{\text{number of ways in which the event occurs}}{\text{total number of possible outcomes}}$
- **empirical** classical probability
  - based on observed data
  - probability of occurrence $= \frac{\text{number of ways in which the event occurs}}{\text{total number of possible outcomes}}$
- **subjective** probability
  - based on a combination of an individual's past experience, personal opinion, and analysis of a particular situation

Note: Both a priori and empirical probability assume all outcomes are equally likely.

Example of a Priori Probability

When randomly selecting a day from the year 2015 what is the probability the day is in January?

$$\text{Probability of Day in January} = \frac{X}{T} = \frac{\text{number of days in January}}{\text{total number of days in 2015}}$$
$$= \frac{\text{31 days in January}}{\text{365 days in 2015}}$$
$$= \frac{31}{365}$$

Example of Empirical Probability

Find the probability of selecting a male taking statistics from the population described in the following table:

	Taking Stats	Not Taking Stats	Total
Male	84	145	229
Female	76	134	210
Total	160	279	439

$$\text{Probability of make taking stats} = \frac{\text{number of males taking stats}}{\text{total number of people}} = \frac{84}{439} = 0.191$$

Subjective Probability

- Subjective probability may differ from person to person
  - A media development team assigns a 60% probability of success to its new ad campaign.
  - The chief media officer of the company is less optimistic and assigns a 40% of success to the same campaign.
- The assignment of a subjective probability is based on a person's experiences, opinions, and analysis of a particular situation.
- Subjective probability is useful in situations when an empirical or a priori probability cannot be computed.

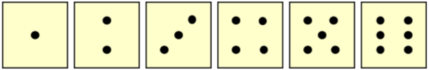
Events

Each possible outcome of a variable is an event.

- **Simple event** (denoted A)
  - An event described by a single characteristic
    (e.g. A day in January from all days in 2015)
- **Complement of an event A** (denoted A')
  - All outcomes that are not part of event A
    (e.g. All days from 2015 that are not in January)
- **Joint event** (denoted $A \cap B$)
  - An event described by two or more characteristics
    (e.g. A day in January that is also a Wednesday from all days in 2015)

Sample Space

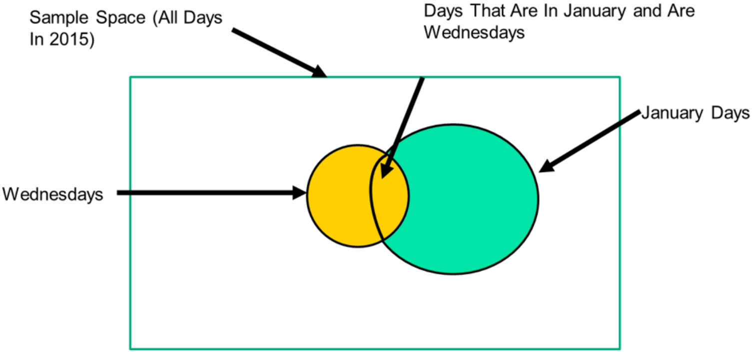
- The Sample Space is the collection of all possible events.
  - e.g. All 6 faces of a die:



  - e.g. All 52 cards of a bridge deck:

Organising and Visualising Events (1 of 2)

- Venn diagram for all days in 2015



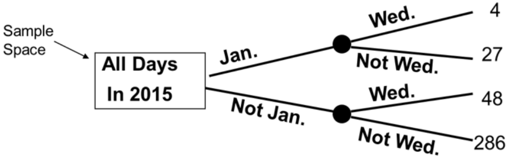
Organising and Visualising Events (2 of 2)

- Contingency tables for all days in 2015

	Jan.	Not Jan.	Total
Wed.	4	48	52
Not Wed.	27	286	313
Total	31	334	365

Total Number of Sample Space Outcomes

- Decision trees



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Marginal Probability

- **Marginal probability** refers to the probability of an event described by a single characteristic.

$$P(A) = P(A \text{ and } B_1) + P(A \text{ and } B_2) + \ldots + P(A \text{ and } B_k)$$

where B_1, B_2, ..., B_k are k mutually exclusive and collectively exhaustive events

	Jan.	Not Jan.	Total
Wed.	4	48	52
Not Wed.	27	286	313
Total	31	334	365

$$P(\text{Wed.}) = \frac{52}{365}$$

$$P(\text{Jan.}) = \frac{31}{365}$$

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Joint Probability

- **Joint probability** refers to the probability of an occurrence of two or more events (joint event).

$$P(A \text{ and } B) = \frac{\text{number of outcomes satisfying A and B}}{\text{total number of elementary outcomes}}$$

	Jan.	Not Jan.	Total
Wed.	**4**	**48**	**52**
Not Wed.	**27**	**286**	**313**
Total	**31**	**334**	**365**

$$P(\text{Not Jan. and Not Wed.}) = \frac{286}{365}$$

$$P(\text{Jan. and Wed.}) = \frac{4}{365}$$

Mutually Exclusive Events

- Two events are **mutually exclusive** if they cannot occur simultaneously.

  e.g. randomly choosing a day from 2015

  A = day in January; B = day in February

  Events A and B are mutually exclusive

Collectively Exhaustive Events

A set of events is **collectively exhaustive** if

- One of the events must occur.
- The set of events covers the entire sample space.

Example

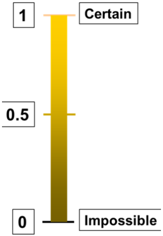
Randomly choose a day from 2015

$$\text{A} = \text{Weekday; B} = \text{Weekend;}$$
$$\text{C} = \text{January; D} = \text{Spring}$$

- Events A, B, C and D are collectively exhaustive (but not mutually exclusive – a weekday can be in January or in Spring)
- Events A and B are collectively exhaustive and also mutually exclusive

Probability Summary So Far

- Probability is the numerical measure of the likelihood that an event will occur.
- The probability of any event must be between 0 and 1, inclusively $0 \leq P(A) \leq 1$ for any event A.
- The sum of the probabilities of all mutually exclusive and collectively exhaustive events is 1.



General Addition Rule

- The **general addition rule** is used to calculate the probability of the joint event A or B.
- The probability of A or B is equal to the probability of A plus the probability of B minus the probability of A and B.

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

- If A and B are mutually exclusive, then P(A and B) = 0, so the rule can be simplified:

$$P(A \text{ or } B) = P(A) + P(B)$$

Example

P(Jan. or Wed.) = P(Jan.) + P(Wed.) - P(Jan. and Wed.)

$$= \frac{31}{365} + \frac{52}{365} - \boxed{\frac{4}{365}} = \frac{79}{365}$$

	Jan.	Not Jan.	Total
Wed.	4	48	52
Not Wed.	27	286	313
Total	31	334	365

Don't count the four Wednesdays in January twice!

Conditional Probability

- A conditional probability is the probability of one event, given that another event has occurred:

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)} \longrightarrow$$ The conditional probability of *A* given that *B* has occurred

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)} \longrightarrow$$ The conditional probability of *B* given that *A* has occurred

Where $P(A \text{ and } B)$ = joint probability of *A* and *B*

$P(A)$ = marginal or simple probability of *A*

$P(B)$ = marginal or simple probability of *B*

Example (1 of 2)

- Of the cars on a used car lot, 70% have air conditioning (AC) and 40% have a GPS. 20% of the cars have both.
- What is the probability that a car has a GPS, given that it has AC?
- i.e. we want to find $P(GPS|AC)$

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Example (2 of 2)

- Given AC, we only consider the top row (70% of the cars). Of these, 20% have a GPS. 20% of 70% is about 28.57%.

	GPS	No GPS	Total
AC	0.2	0.5	0.7
No AC	0.2	0.1	0.3
Total	0.4	0.6	1.0

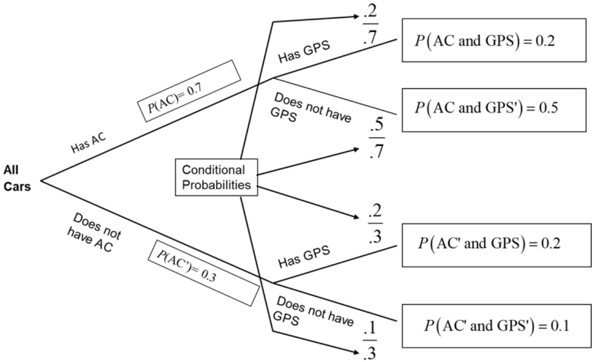
$$P(\text{GPS}|\text{AC}) = \frac{P(\text{GPS and AC})}{P(\text{AC})} = \frac{0.2}{0.7} = 0.2857$$

Decision Trees

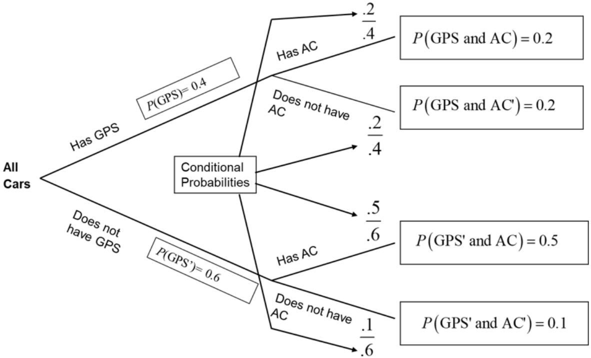
- A **decision tree** (or **tree diagram**) is a graphical representation of simple and joint possibilities as vertices of a tree.
- A decision tree:
  - is an alternative to contingency tables
  - allows sequential events to be graphed
  - allows calculation of joint probabilities by multiplying respective branch probabilities

Using Decision Trees (1 of 2)

Given AC or no AC:



Given GPS or no GPS:



Statistical Independence

- Events A and B are **independent** when the probability of one event is not affected by the fact that the other event has occurred.
- Two events are independent if and only if:

$$P(A|B) = P(A) \quad \text{or} \quad P(B|A) = P(B)$$

Multiplication Rules (1 of 2)

- The **general multiplication rule** is used to calculate the probability of the joint event A and B.
- The probability of A and B is equal to the probability of A given B times the probability of B or the probability of B given A times the probability of A.

$$P(A \text{ and } B) = P(A|B)P(B) = P(B|A)P(A)$$

Multiplication Rules (2 of 2)

- The **multiplication rule for independent events** is used to calculate the probability of the joint event A and B when A and B are independent.
- If A and B are statistically independent, the probability of A and B is equal to the probability of A times the probability of B.

$$P(A \text{ and } B) = P(A)P(B)$$

You Do It

Alex has developed a stock screen - a set of criteria for selecting stocks. He has decided that S&P 500 will be his investment universe. Also, he believes that the criteria are independent of each other and capture different aspects of the selection problem. (Hint: S&P 500 index contains stocks of 500 large-cap U.S. companies)

Criterion	Fraction of S&P 500 stocks meeting criterion
A	0.45
B	0.57
C	0.50
D	0.25
E	0.67

How many stocks will pass the screen developed by Alex?

A. 10
B. 11
C. 12

Bayes' Theorem (1 of 2)

- **Bayes' Theorem** is used to revise previously calculated probabilities based on new information.
- Developed by Thomas Bayes in the 18th Century.
- It is an extension of conditional probability.
- Need to identify:
    - Prior probabilities: $P(S_i)$
    - Conditional probabilities: $P(F|S_i)$
- Then we can calculate:
    - Joint probabilities: $P(F|S_i)P(S_i)$
    - Revised probabilities: $P(S_i|F)$

Bayes' Theorem (2 of 2)

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + ... + P(A|B_k)P(B_k)}$$

where

B_i = ith event of k mutually exclusive and collectively exhaustive events

A = new event that might impact $P(B_i)$

Example (1 of 3)

- A drilling company has estimated a 40% chance of striking oil for their new well.
- A detailed test has been scheduled for more information. Historically, 60% of successful wells have had detailed tests, and 20% of unsuccessful wells have had detailed tests.
- Given that this well has been scheduled for a detailed test, what is the probability that the well will be successful?

Example (2 of 3)

- Let S = successful well
  U = unsuccessful well
- Prior probabilities
  - $P(S) = 0.4$, $P(U) = 0.6$
- Conditional probabilities
  - Define the detailed test event as D
  - $P(D|S) = 0.6$, $P(D|U) = 0.2$
- Goal is to find $P(S|D)$

Example (3 of 3)

$$
\begin{aligned}
P(S|D) &= \frac{P(D|S)P(S)}{P(D|S)P(S) + P(D|U)P(U)} \\
&= \frac{(0.6)(0.4)}{(0.6)(0.4) + (0.2)(0.6)} \\
&= \frac{0.24}{0.24 + 0.12} \\
&= 0.667
\end{aligned}
$$

So the revised probability of success, given that this well has been scheduled for a detailed test, is 0.667.

Counting Rules Are Often Useful in Computing Probabilities

- In many cases, there are a large number of possible outcomes.
- Counting rules can be used in these cases to help compute probabilities.

Counting rules (1 of 5)

- **Counting rules** are rules for counting the number of possible outcomes.
- Counting Rule 1
  - If any one of k different mutually exclusive and collectively exhaustive events can occur on each of n trials, the number of possible outcomes is equal to k^n.
  - Example
    – If you roll a fair die 3 times then there are $6^3 = 216$ possible outcomes.

Counting rules (2 of 5)

- Counting Rule 2
  - If there are k_1 events on the first trial, k_2 events on the second trial, ... and k_n events on the nth trial, the number of possible outcomes is $(k_1)(k_2)\cdots(k_n)$.
  - Example
    – You want to go to a park, eat at a restaurant and see a movie. There are 3 parks, 4 restaurants and 6 movie choices. How many different possible combinations are there?
    – Answer: $3 \cdot 4 \cdot 6 = 72$ possible outcomes

Counting rules (3 of 5)

- Counting Rule 3
  - The number of ways that n items can be arranged in order is $n! = (n)(n-1)...(1)$.
  - Example
    – You have five books to put on a bookshelf. How many different ways can these books be placed on the shelf?
    – Answer: $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ different possibilities

Counting rules (4 of 5)

- Counting Rule 4
  - Permutations: The number of ways of arranging X objects selected from n objects in order is

$$P(n, X) = \frac{n!}{(n-X)!}$$

  - Example
    – You have five books and are going to put three on a bookshelf. How many different ways can the books be ordered on the bookshelf?
    – Answer: $P(n, X) = \frac{n!}{(n-X)!} = \frac{5!}{(5-3)!} = \frac{120}{2} = 60$ different permutations

Counting rules (5 of 5)

- Counting Rule 5
  - Combinations: The number of ways of selecting X objects from n objects, irrespective of order, is equal to:

$$\binom{n}{X} = \frac{n!}{X!(n-X)!}$$

  - Example
    – You have five books and are going to select three are to read. How many different combinations are there, ignoring the order in which they are selected?
    – Answer: $\binom{n}{X} = \frac{n!}{X!(n-X)!} = \frac{5!}{3!(5-3)!} = \frac{120}{6 \cdot 2} = 10$ different combinations

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You Do It

You have applied for a Masters degree program at Curtin University and University of Queensland (UQ). The probability that you will get a call from Curtin University is 0.70 and from UQ is 0.50. Assuming that these are mutually exclusive and exhaustive events, what is the probability that you will get a call from at least one university?

A. 0.20
B. 0.85
C. 0.35